Optimum Serpentine Acceleration in Scaling FFAG

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Abstract

Serpentine acceleration is typified by fixed radio frequency, fixed magnetic field and a near (but not) isochronous lattice, radial motion of the orbit, and two or more reversals of the motion in RF phase. This was discovered in 2003 for linear non-scaling FFAGs in the relativistic regime. In 2012, Kyoto University School of Engineering showed that serpentine acceleration is possible also in scaling FFAGs and may span the non-relativistic to relativistic regime. As a function of two key parameters, field index and synchronous energy, this paper shows how to optimize the extraction energy and the voltage per turn for the scaling case. Optimization is difficult, and typically leads to poor performance: either extreme voltage or small acceleration range. Nevertheless, designs with credible acceleration parameters can be obtained; and indicative examples are presented herein.

Introduction

In the scaling FFAG, the magnet field has the form:

$$B_z(R, z = 0) = (R/R_0)^k$$

where $k > 0$ is the field index. R_0 is a reference radius. The subscript s shall denote synchronous value. The general orbit radius is given by

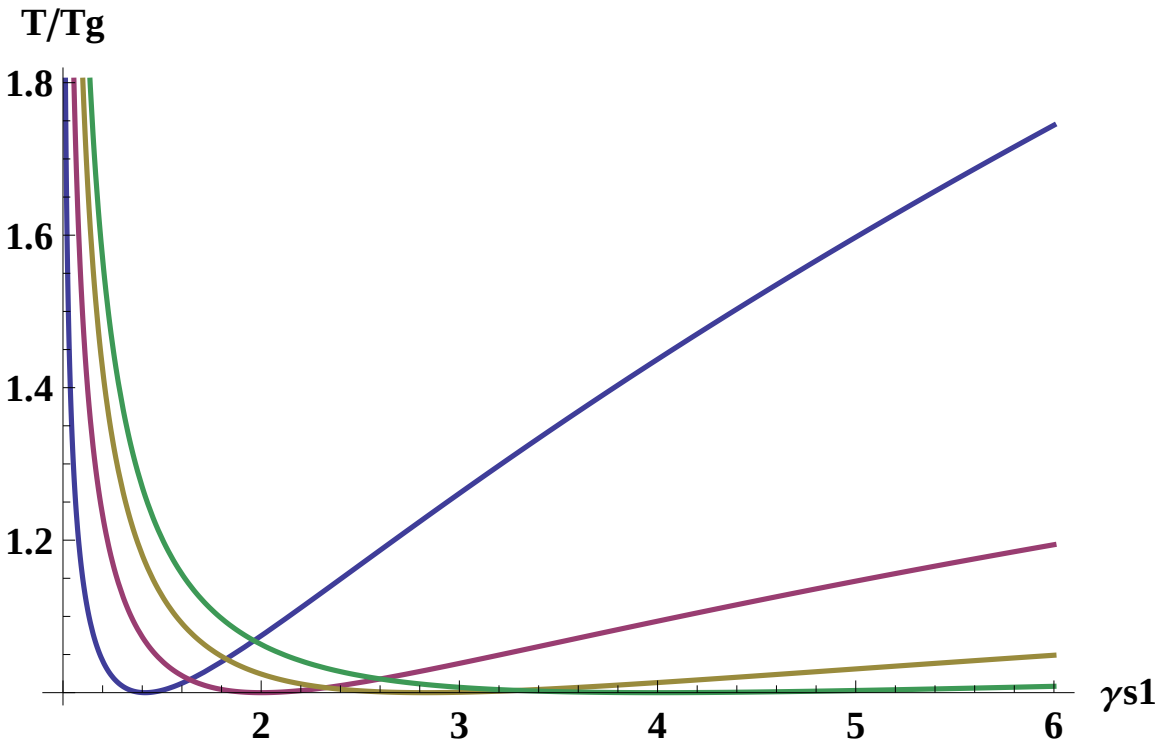
$$R/R_s = (P/P_s)^\alpha$$

where $\alpha = 1/(1 + k) < 1$ is solely a property of the lattice. It follows that revolution period T as a function of E, P is given by

$$T/T_s = (E/E_s)(P/P_s)^{(-1+\alpha)} = (\beta_s/\beta)[(\beta\gamma)/(\beta_s\gamma_s)]^\alpha \ .$$

Here γ is the relativistic kinematic factor, $E = E_0\gamma$ and $E_0 = m_0c^2$ is the rest mass energy. We define $T \equiv T(\gamma)$, $T_s \equiv T(\gamma_s)$ and $T_t \equiv T(\gamma_t)$ where $E_s = E_0\gamma_s$ is a synchronous energy and $E_t = E_0\gamma_t$ is the transition energy. One may eliminate $\beta = v/c$ in favour of γ.

Orbit Revolution Period



Period versus energy (γ) for $\alpha = 1/2$ (blue), 1/4 (red), 1/8 (yellow), 1/16 (green).

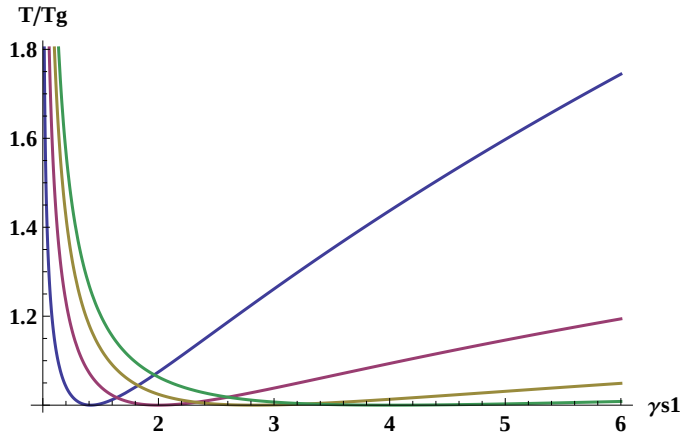
Two Synchronous Energies

The curves are ”U” or ”V”-shaped. $\gamma(T)$ is a double valued function: to each value of T belongs two values of γ.

Each curve have a minimum which defines the transition energy. Solving $\partial(T/T_s)/\partial\gamma = 0$, one finds $\gamma_t = 1/\sqrt{\alpha}$.

For brevity, let $\gamma_{s1} \equiv \gamma_1$ and $\gamma_{s2} \equiv \gamma_2$ be two energies having the same revolution period; there is a continuum of such doublets. We shall adhere to the convention that $\gamma_1 < \gamma_t < \gamma_2$. A certain doublet is chosen to be the synchronous reference when we set the radio frequency (RF) to be co-periodic with the orbit period $T(\gamma_1) = T(\gamma_2)$. Once this is chosen E_1, E_2 become fixed points of the motion. Both values of the synchronous E_s are equally valid!

It is a little arbitrary, but we choose to work with the lower E_{s1} because it exists in the narrow range $1 < \gamma_{s1} < \gamma_t$.

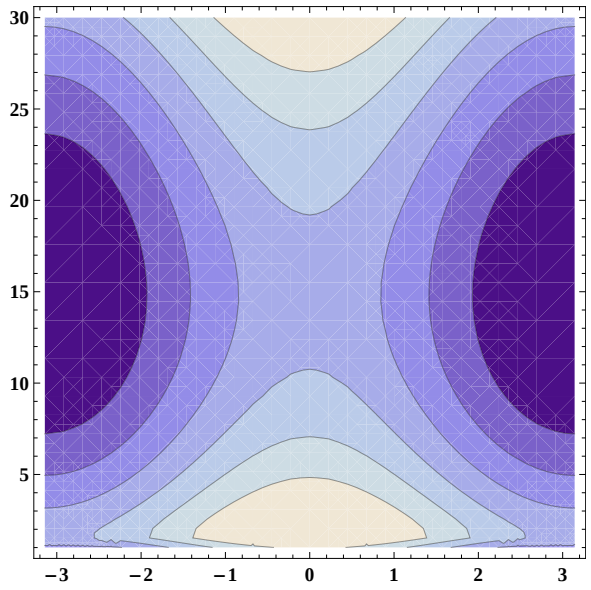


The general features of the T/T_g curves are a very steep rise as $\gamma \to 1$, and a long slow ramp for $\gamma \gg \gamma_t$. When selecting reference doublets, this has the consequence that as $\gamma_1 \to 1$, so $\gamma_2 \to \infty$. Thus the range of acceleration is unbounded. But this range is illusory, and corresponds to a linac-like regime with prodigious voltage requirement.

Hamiltonian

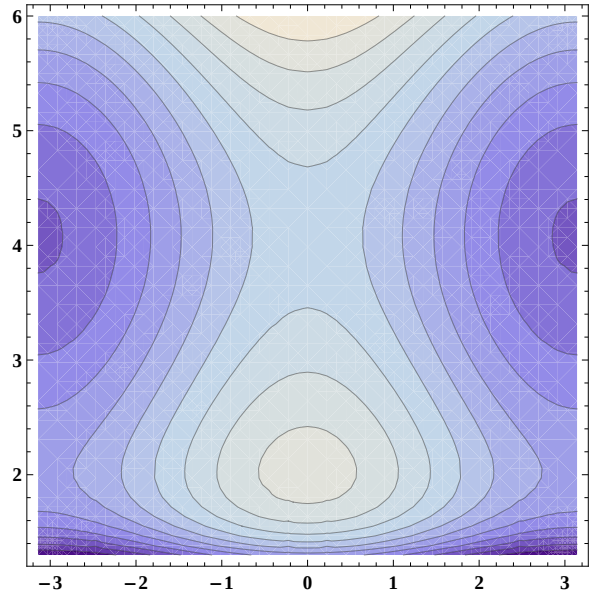
$$H(E,P,\phi) \equiv -Eh + h(PP_s)\frac{(P/P_s)^{\alpha}}{E_s(1+\alpha)} + \frac{eV\cos\phi}{2\pi}\,. \qquad (1)$$

Because of the FFAG scaling property, the Hamiltonian is invariant whether we use E_1 or E_2 for the synchronous energy. We set h times their common revolution frequency equal to the radio frequency. These two energies are either side of transition; so, during acceleration, the direction of phase slip for the entire beam reverses twice.



Case 1:

$(\alpha, \gamma_s) = (0.09357, 1.650)$



Case 3:

$(\alpha, \gamma_s) = (0.1371, 2.427)$

Phase space contours: energy (γ) versus RF phase (ϕ).

Serpentine acceleration in the S-shape channel between two RF buckets offset in energy can be greater than the range (bottom to top) within a single RF bucket.

Acceleration Range

The range is the sum of three phase space arcs: (i) from the injection energy E_i to the first synchronous energy E_1; (ii) a path between E_1 and E_2; (iii) from the second synchronous energy to the extraction energy E_x.

The extraction energy is obtained by equating $H(E_x, P_x, \pi) = H(E_2, P_2, 0)$, writing $E_x = E_2 + \delta E_x$, and solving for the increment

$$\delta E_x^2 \approx \frac{2V/\pi h}{+1/E_2 - E_2/P_2^2(1-\alpha)}$$

The injection energy is obtained by equating $H(E_i, P_i, 0) = H(E_1, P_1, \pi)$, writing $E_i = E_1 - \delta E_i$, and solving for the increment

$$\delta E_i^2 \approx \frac{2V/\pi h}{-1/E_1 + E_1/P_1^2(1-\alpha)}$$

The energy range of the machine is

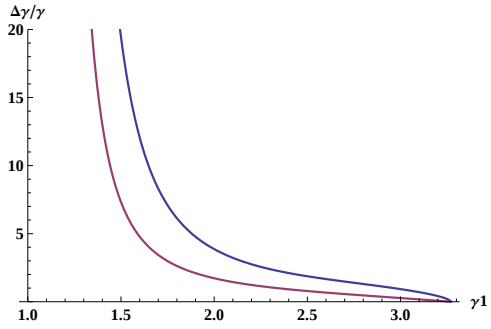
$$\Delta E = (E_x - E_i) \approx (E_2 - E_1) + \delta E_i + \delta E_x \sim 2E_2$$

which is expressible solely in terms of E_1, E_2. But E_2 is expressible in terms of E_1: $E_2(E_1)$ is the solution of $T(E_1) = T(E_2)$. Hence there is an expression for the energy range in terms of E_1, V, α.

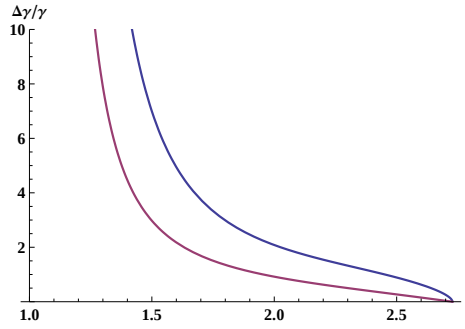
Typically $\delta E_i \ll \delta E_x$: $\delta E_i \sim P_1 c\sqrt{2}$ and $\delta E_x \sim E_2\sqrt{2}$.

Typically, $\Delta E = (E_x - E_i) \sim 2E_2$.

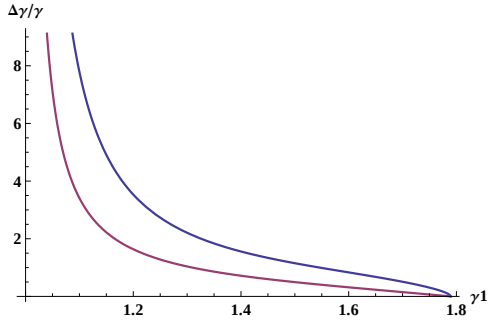
The total acceleration range ΔE (blue) and the contribution from the fixed points $(E_2 - E_1)$ (red) for a particular α as a function of γ_{s1}. The quantities are normalized by the transition energy. As $\gamma_s \to 1$ the range becomes unbounded; and as $\gamma_s \to \gamma_t$ the range shrinks to zero.



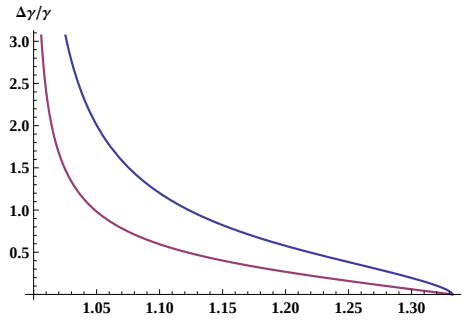
Case 1.



Case 2.



Case 4.



Case 5.

Minimum Voltage

The condition to connect the two fixed points E_1 and E_2 by a phase space path of zero width is obtained by equating the two Hamiltonians $H(E_1, P_1, \pi) = H(E_2, P_2, 0)$ and solving for voltage per turn:

$$\frac{eV_0}{\pi h} = (E_2 - E_1) + \frac{(E_2 P_1^2 - E_1 P_2^2)}{E_1 E_2 (1 + \alpha)} \,. \tag{2}$$

Evidently, one prefers low harmonic number. Eliminating the momenta leads to

$$\frac{eV}{E_0} = \pi h \frac{(\gamma_2 - \gamma_1)(\gamma_1 \gamma_2 \alpha - 1)}{\gamma_1 \gamma_2 (1 + \alpha)} \,. \tag{3}$$

This is a very significant relation.

linac and ring like regimes

If $\alpha\gamma_1\gamma_2 \gg 1$ this corresponds to acceleration in a linac-like regime (case 1) in which $\Delta\gamma/\gamma_t \gg 1$ and

$$eV_0/E_0 \to (\gamma_2 - \gamma_1)\alpha\pi h/(1+\alpha)$$

This is a very few turn acceleration regime, and there is little point employing an FFAG ring unless the particles are very short lived. The required voltage is prodigious: order the rest mass energy per turn; this may be acceptable for leptons (e.g. 0.5 MeV for e) but not for hadrons (e.g. 1 GeV for p).

Contrastingly, if $\alpha\gamma_1\gamma_2 \to 1$ then $V \to 0$. In principle, this implies $\Delta E/eV \to \infty$; but $\Delta\gamma/\gamma_t \to 0$. This corresponds to acceleration in a ring-like regime (case 3), with tiny voltage and many turns but with a small range.

Minimum voltage

$$\frac{eV}{E_0} = \pi h \frac{(\gamma_2 - \gamma_1)(\gamma_1\gamma_2\alpha - 1)}{\gamma_1\gamma_2(1+\alpha)} \ . \tag{4}$$

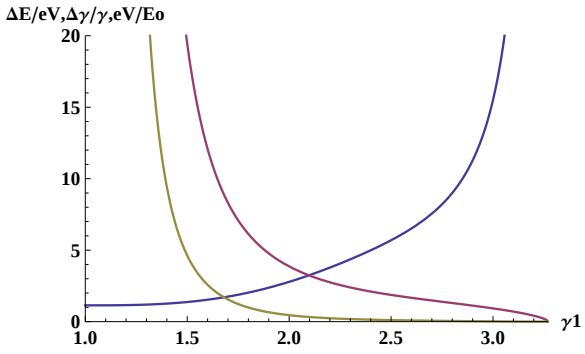
By fine tuning of parameters, this feature may be exploited to give a limited multi-turn acceleration (cases 2,4,5).

$\alpha\gamma_1\gamma_2 = 1$ has the single solution is $\gamma_1\gamma_2 = \gamma_t$. For all other values such that $T(\gamma_1) = T(\gamma_2)$, $\alpha\gamma_1\gamma_2 > 1$ and rises progressively rapidly because γ_2 increases more quickly than γ_1 falls. Clearly, it is an advantage to use small α.

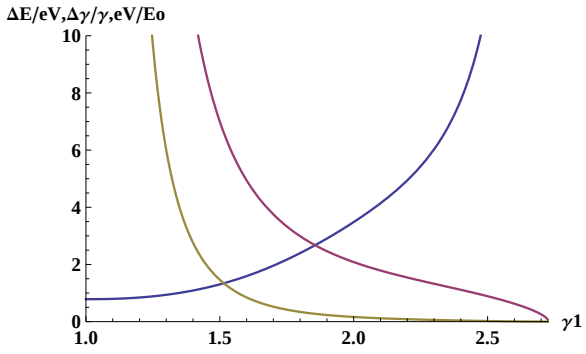
Optimization

Our task would appear to be to maximize the acceleration range for a given value of the voltage per turn V.

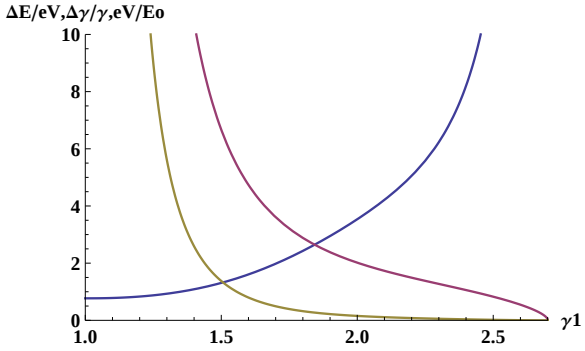
Figure shows the normalized range $\Delta E/E_t$ (red), voltage eV_0/E_0 (yellow), and $\Delta E/eV_0$ (blue) which is roughly the number of turns, and as function of γ_{s1}. While $\Delta E/eV$ rises, the acceleration range falls dramatically; the voltage per turn falls even more precipitously. These behaviours are common to all values of α.



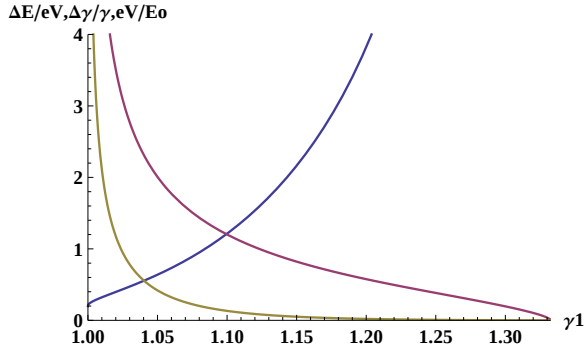
Case 1.



Case 2.



Case 3.



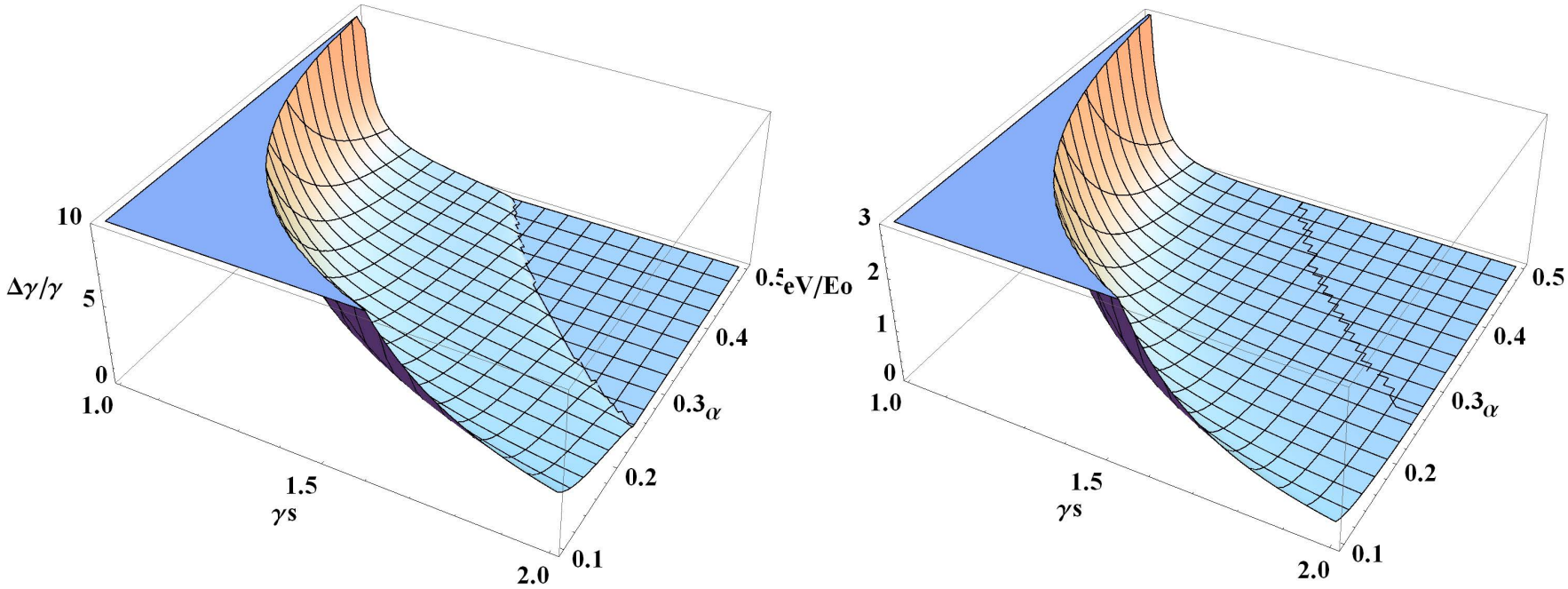
Case 5.

The minimum voltage per turn is essentially the product of range and a quantity that diminishes as $\gamma_{s1} \rightarrow \gamma_t$. This has two consequences for the combination $\Delta E/eV_0$: (i) it is independent of range; and (ii) it rises as the range diminishes. Contrary to expectations, $\Delta E/eV$ is not a suitable figure of merit upon which to base optimization.

So we must apply to ΔE and eV_0 directly as the basis for optimization.

We know that $\gamma_{s1} \rightarrow 1$ (large range, large voltage, few turns) and $\gamma_{s1} \rightarrow \gamma_t$ (small range, tiny voltage, many turns) are both poor choices for the synchronous energy.

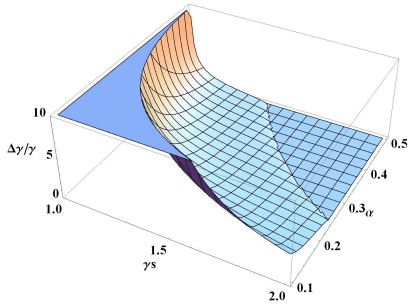
But one may speculate that useful working points exit between these extremes. Our approach is to take combinations $[\gamma_1, \gamma_2]$ which satisfy $T(\gamma_1) = T(\gamma_2)$ exactly, and roughly satisfy $\gamma_1 \gamma_2 \approx \gamma_t^2$. The optimization amounts to scanning α, γ_{s1}.



Normalized range (left) and required voltage (right) as function of α, γ_{s1}. Range of $\alpha = [0.1, 0.5]$.

Figure shows that maximizing the energy range and minimizing the voltage are contradictory efforts. Thus one must choose, for given index α, either the range and accept the voltage, or place a limit on voltage per turn and accept the energy range.

Alternatively, for given range and voltage values one may search for the (α, γ_s) combination that leads to the largest value of α (i.e. smallest value of k) and hence the easiest-to-realize magnetic lattice.



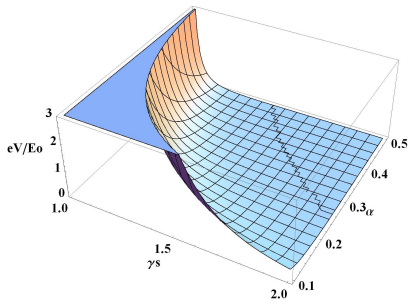


Figure exemplifies the challenge. Let ρ, ν be target values. Optimization corresponds to finding the intersection of the two surfaces: $(\Delta\gamma/\gamma_t)/\rho \geq 1$ and $(eV_0/E_0)/\nu \leq 1$ in the (α, γ_{s1}) plane, which leads to a curve.

Introducing the objective of greatest α leads to a single point and the condition $(\Delta\gamma/\gamma_t)/\rho = (eV_0/E_0)/\nu$ to be solved for (α, γ_{s1}).

Examples

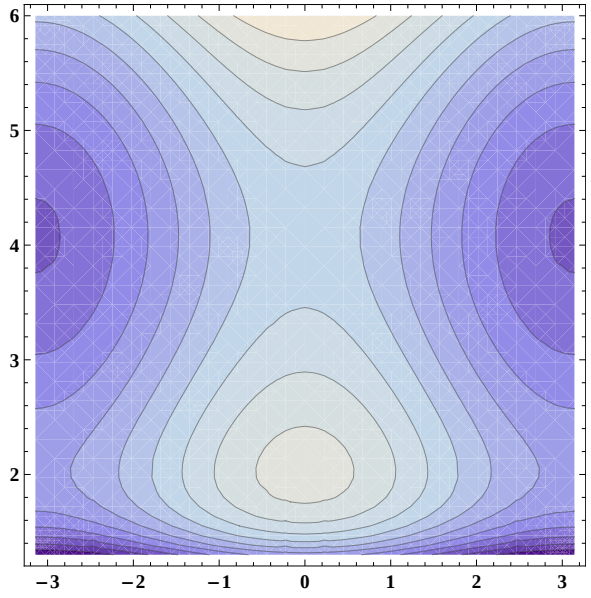
We present seven examples, each with different design objectives: (ρ, ν). The first case is linac-like, with large range and voltage. The third case is ring-like, with small voltage and many turns. The fifth case is a toy accelerator that spans the Newtonian to relativistic region. The second and fourth cases are intermediate with similar number of turns, but with opposing tendency of α and $\delta\gamma \equiv eV_0/E_0$. The sixth case is that of Kyoto University POP 8 MeV electron FFAG. The seventh is a competitor with more relaxed field index and voltage per turn.

Examples

#	$\Delta\gamma/\gamma_t$	$\delta\gamma$	turn	α	k	γ_{inj}	γ_{s1}	γ_t	γ_{s2}	$\Delta\gamma$
1	10.	2.0	16.35	0.0936	9.688	1.0	1.650	3.269	14.77	32.38
2	2.0	0.150	36.37	0.1344	6.44	1.306	2.025	2.728	4.419	5.462
3	1.0	0.030	90.03	0.1371	6.3	1.768	2.427	2.701	3.304	2.70
4	1.0	0.050	35.79	0.3123	2.2	1.208	1.546	1.789	2.279	1.789
5	0.75	0.040	25.0	0.5632	0.775	1.027	1.163	1.333	1.647	1.000
6	6.28	1.38	11.3	0.163	5.13	1.00	1.41	2.48	8.13	15.5
7	4.0	0.70	13.3	0.184	4.42	1.00	1.46	2.33	5.65	9.3

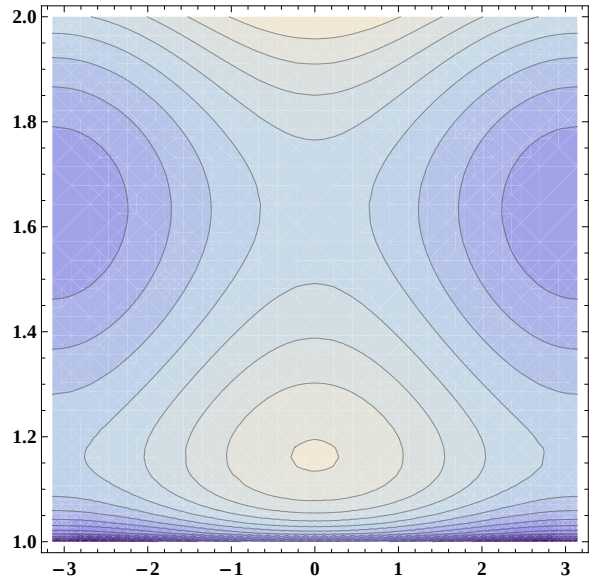
Turn $\approx \Delta E/eV_0$

Phase space contours: energy (γ) versus RF phase (ϕ).



Case 2:

$(\alpha, \gamma_s) = (0.134416, 2.02489)$



Case 5:

$(\alpha, \gamma_s) = (0.563233, 1.16276)$

Conclusion

The scaling FFAG proves to be a versatile platform for exploiting serpentine acceleration. However, the performance is generally poor: either the voltages are large and the turns are few, or the voltages and acceleration range are small. In either case, other accelerator types (linac and cyclotron, respectively) would be more effective.

Nevertheless, careful optimization can produce intermediates cases with credible parameters that have the appeal of acceleration over the Newtonian ($\gamma \approx 1$) to relativistic regime ($\gamma \gg 1$).

Note, these conclusions do not apply to scaling FFAGs with swept RF; they are a class distinct from the considerations above.

References

- S. Koscielniak and C.J. Johnstone: “Longitudinal Dynamics in an FFAG Accelerator Under Conditions of Rapid Acceleration and Fixed, High, RF”, Proc. 2003 Particle Accelerator Conference, Seattle, WA, USA.

- NIM E. Yamakawa et al: “Serpentine Acceleration in Zero-Chromatic FFAG Accelerators”, Nuclear Instruments and Methods A, Volume 716, 11 July 2013, Pages 4653.